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NOTE ON THE SIMPLE DIFFERENCE EQUATION.

By J. H. M. Wedderburn.

Boole on his Calculus of Finite Differences derives the Euler-Maclaurin sum formula from the expansion of the operator $\Delta^{-1} = (e^D - 1)^{-1}$ in a series of positive powers of D so obtaining the well known result

$$(1)\quad \Delta^{-1}g(x) = \int^x g(x)dx - \tfrac{1}{2}g(x) + \sum_1^\infty (-1)^n \frac{B_{2n+1}}{(2n+2)!}\frac{d^{2n+1}g(x)}{dx^{2n+1}} + \omega(x),$$

where $\omega(x)$ is an arbitrary periodic function of period unity. This series is usually divergent and is besides only applicable to functions which possess derivatives of every order.

If instead of expanding $(e^D - 1)^{-1}$ in powers of D we use the series

$$(2)\qquad \frac{1}{e^D - 1} = \frac{1}{D} - \frac{1}{2} + \sum_1^\infty \left[\frac{1}{D + 2n\pi i} - \frac{1}{D - 2n\pi i}\right],$$

we obtain another form of the solution of the equation $\Delta f(x) = g(x)$, namely

$$(3)\quad f(x) = -\tfrac{1}{2}g(x) + \int^x g(t)dt + 2\sum_1^\infty \int^x \cos 2\pi n(x - t)g(t)dt + \omega(x),$$

the lower limits being arbitrary constants whose imaginary parts are all equal and the path of integration a line parallel to the real axis.

The series (3) has this advantage over (1) that it is usually convergent, the conditions of convergence being similar to those of a Fourier series. It is however in general discontinuous; for, when the lower limits are all made equal to zero, then if

$$\begin{array}{rl}
x = 0, & f(0) = -\tfrac{1}{2}g(0), \\
0 \leqq x \leqq 1, & f(x) = 0, \\
x = 1, & f(1) = \tfrac{1}{2}g(0), \\
1 \leqq x \leqq 2, & f(x) = g(x - 1), \\
\cdot\quad\cdot\quad\cdot\quad\cdot & \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\
n \leqq x \leqq n + 1, & f(x) = g(x - 1) + g(x - 2) + \cdots g(x - n).
\end{array}$$

As the derivation of the series (3) is purely formal, it is necessary to verify directly that it does satisfy the equation $\Delta f(x) = g(x)$. Forming

$\Delta f(x)$ we have

$$\Delta f(x) = -\tfrac{1}{2}[g(x+1) - g(x)] + \int_x^{x+1} g(t)dt + 2\sum_1^\infty \int_x^{x+1} \cos 2\pi n(x-t)g(t)dt$$

$$= -\tfrac{1}{2}[g(x+1) - g(x)] + \tfrac{1}{2}[g(x+1) + g(x)] = g(x),$$

since the series

$$\int_x^{x+1} g(t)dt + 2\sum_1^\infty \int_x^{x+1} \cos 2\pi n(x-t)g(t)dt$$

is the value of the Fourier series for $g(x)$ at the end of the range of integration. Hence (3) gives a value of $\Delta^{-1}g(x)$ when $g(x)$ is a continuous function capable of being represented by a Fourier series.

This solution is of course not essentially different from the sum formula of Poisson: it is also closely related to the Guichard integral from which it may be derived.

2. As an illustration of the use of our form of the solution we shall now derive from it the well known asymptotic series for $\log \Gamma(x)$ which is a solution of the equation

$$\Delta f(x) = \log x. \tag{4}$$

Forming the series for $f(x)$, we obtain as a solution of (4)

$$f_1(x) = -\tfrac{1}{2}\log x + \int_1^x \log t dt + 2\sum_1^\infty \int_1^x \cos 2\pi n(x-t) \log t dt$$

$$= (x - \tfrac{1}{2})\log x - x + 1 + 2\sum_1^\infty - \left[\frac{\sin 2\pi n(x-t)}{2\pi n}\log t\right]_1^x$$

$$+ 2\sum_1^\infty \int_1^x \frac{\sin 2\pi n(x-t)}{2\pi n t}$$

$$= (x - \tfrac{1}{2})\log x - x + 1 + 2\sum_1^\infty \int_1^x \frac{\sin 2\pi n(x-t)}{2\pi n t}dt.$$

If we now change the lower limit of integration from 1 to ∞, the value of the series is only changed by a periodic function of period unity, so that

$$f_2(x) = (x - \tfrac{1}{2})\log x - x + k - 2\sum_1^\infty \int_x^\infty \frac{\sin 2\pi n(x-t)}{2\pi n t}dt$$

$$= (x - \tfrac{1}{2})\log x - x + k + 2\sum_1^\infty \frac{1}{(2\pi n)^2}x^{-1} + 2\sum_1^\infty \int_x^\infty \frac{\cos 2\pi n(x-t)}{(2\pi n t)^2 t^2}dt,$$

where k is a constant, is a solution provided the series used are convergent.

When x is real and positive,

$$| S | = \left| \sum_1^\infty \int_x^\infty \frac{\cos 2\pi n(x - t)}{(2\pi n)^2 t^2} dt \right| \leqq \sum_1^\infty \int_x^\infty \frac{dt}{(2\pi n)^2 t^2} = \frac{1}{x} \sum_1^\infty \frac{1}{(2\pi n)^2},$$

while, if $x = \xi + \eta i$, $t = \alpha + \eta i$, $(\eta \neq 0)$,

$$| S | \leqq \sum_1^\infty \int_\xi^\infty \frac{d\alpha}{(2\pi n)^2(\alpha^2 + \eta^2)} \leqq \frac{\pi}{|\eta|} \sum_1^\infty \frac{1}{(2\pi n)^2}.$$

The series therefore converges and

$$f_2(x) \sim k + (x - \tfrac{1}{2}) \log x - x + \frac{2}{x} \sum_1^\infty \frac{1}{(2\pi n)^2}$$

so long as x does not lie on the negative half of the real axis. It is easily shown by the usual methods that k must equal $\frac{1}{2} \log 2\pi$ in order that $f_2(x)$ may coincide with $\log \Gamma(x)$.

3. Another solution of the simple difference equation may be obtained by means of the integral equation

$$\int^x g(t) dt = \int_x^{x+1} f(t) dt, \tag{7}$$

which on differentiation leads to

$$g(x) = f(x + 1) - f(x).$$

As is suggested by the solution of the Fredholm equation, we consider first the homogeneous equation

$$u(x) = \lambda \int_x^{x+1} u(t) dt = \lambda \int_a^\infty K(x, t) u(t) dt, \tag{8}$$

where the kernel $K(x, t)$ is defined by

$$\begin{aligned} K(x, t) &= 0, \quad \alpha \leqq t < x \\ &= 1, \quad x \leqq t \leqq x + 1 \\ &= 0, \quad x + 1 < t, \end{aligned}$$

α being a fixed constant. This kernel has a continuous spectrum: for if in (8) we substitute

$$u(x) = e^{kx}, \qquad (k \neq 2n\pi i) \tag{9}$$

we get, after a simple reduction,

$$\lambda = \frac{k}{e^k - 1}, \tag{10}$$

so that λ may have any finite value.

The theory of integral equations therefore suggests the expansion of $\int^x g(t)dt$ in a series of the form

$$\int^x g(t)dt = \Sigma a_i e^{k_i x}. \tag{11}$$

If an expansion of this form exists, a solution of (7) is

$$f(x) = \Sigma a_i \frac{k_i}{e^{k_i} - 1} e^{k_i x}, \tag{12}$$

provided this series is convergent.

If $g(x)$ can be expanded in a uniformly convergent series

$$g(x) = \Sigma b_i e^{k_i x}, \tag{13}$$

then $b_i = a_i k_i$, and we may set for $f(x)$

$$f(x) = \Sigma \frac{b_i}{e^{k_i} - 1} e^{k_i x} + \omega(x). \tag{14}$$

When $g(x)$ is a real function which satisfies Dirichlet's conditions, an expansion of the form (13) always exists. In particular we have*

$$g(x) = \Sigma \frac{e^{a\sigma i}(\sigma i + h)}{\sin 2a\sigma - 2a\sigma} \sin a\sigma \cdot e^{\sigma i x} \int_{-a}^{a} e^{-\sigma i t} g(t)dt, \tag{15}$$

where the summation extends over all the roots of

$$\sigma + h \tan a\sigma = 0, \tag{16}$$

and $-a < x < a$.

None of the roots of (16) make $e^{\sigma i} - 1 = 0$, and as σ approaches infinity, σ approaches the value $(2n + 1)\pi/2a$, so that if a is an integer, $e^{\sigma i} - 1$ remains greater than some finite quantity. Hence the form of $f(x)$ given by (14) is convergent and a solution of (7) is therefore given by

$$\begin{aligned} f(x) &= \frac{hx}{2(1+ah)} \int_{-a}^{a} g(t)dt + \Sigma \frac{e^{a\sigma i}(\sigma i + h) \sin a\sigma}{(\sin 2a\sigma - 2a\sigma)(e^{\sigma i} - 1)} \int_{-a}^{a} e^{\sigma i(x-t)} g(t)dt + \omega(x) \\ &= \frac{hx}{2(1+ah)} \int_{-a}^{a} g(t)dt + \Sigma \frac{\sigma}{2a\sigma - \sin 2a\sigma} \int_{-a}^{a} g(t) \frac{\sin \sigma(x - t - \frac{1}{2})}{\sin \frac{1}{2}\sigma} dt + \omega(x), \end{aligned}$$

where the second summation extends only over positive values of σ.

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* See for instance Picard, Traité d'Analyse, 2e ed., t. 2, p. 192.